NASA Contractor Report 181761

ICASE REPORT NO. 88-73

# ICASE

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## PART 2: THEORETICAL JUSTIFICATION

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Contract Nos. NAS1-18107 and NAS1-18605
December 1988

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NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association



National Aeronautics and
Space Administration

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# Spectral Element Multigrid
# Part 2 : Theoretical Justification*

by

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**Abstract** : We analyze here a multigrid algorithm used for solving iteratively the algebraic system resulting from the approximation of a second order problem by spectral or spectral element methods. The analysis, performed here in the one dimensional case, justifies the good smoothing properties of the Jacobi preconditioner that has been presented in the part 1 of this paper.



* This research was supported in part by AFOSR Grant No. 850303 while the first author was visiting the Division of Applied Mathematics at Brown University. Additional support was provided by the National Aeronautics and Space Administration under NASA Contract Nos. NAS1-18107 and NAS1-18605 while the first author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665.

## I. Introduction.

Spectral element methods are high order methods that combine the flexibility of finite element methods with the "infinite order accuracy" of spectral methods. The domain of computation is decomposed into some subdomains - the elements - (generally these are deformed parallelotopes), and the exact solution is approximated by a piecewise polynomial of high degree. The spectral element method differs from other spectral methods using domain decomposition techniques (patching methods) by the way the matching conditions are handled. These are, like in the finite element method, implicitly taken into account by the variationnal statement of the discrete problem. This allows for more flexibility with no loss of the spectral accuracy ( see, e.g., [P], [M.P.] and [F]). When the algebraic equations resulting from this kind of discretization are obtained, the problem that remains is to solve, in an efficient way, the algebraic system.

The interest of domain decomposition technique is to fraction the computational task so as to yield smaller problems and to use parallel computers for instance. If the value of the approximate solution were known on the various interfaces, the problem would be very simple since it would results into the resolution of as many disconnected problems as the number of elements. The main difficulty is that this value is not known; hence a technique known as the iteration per subdomains has been proposed in the literature [F.Q.Z.] [Q.S1] to discover this value iteratively. Another approach is to try to invert the whole problem by not working iteratively on each subdomain. This method, [M.P.] [F.R.D.P.] [R], consists in reducing iteratively the residue at the same time on every subdomain. The global method used can be based on a conjugate gradient algorithm or another iterative procedure.

In the first part of this paper, E. M. Ronquist and A. T. Patera [R.P.] have presented some results concerning a new multigrid method for the resolution of the algebraic system resulting from the approximation of a second order P.D.E. by spectral or spectral element method in the one-dimensional domains. The very simple idea of using the Jacobi preconditioner as a smoother for the iterative multigrid algorithm appears to be a very good one. Indeed, the numerical properties of this smoother seems to surpass all expectations; the reduction rate of each V-cycle appears numerically to be independent of the discretization parameters.

When iterative techniques are used, it is important to understand why these methods converge in order to foresee the generalization and the ability of the methods to be adapted to more than test problems. Here we propose an analysis of this phenomenon and provide the justification of these very good properties. Many general convergence proofs exist in the literature for the numerical analysis of the multigrid technique; among them let us cite [M.M.], [B.D.]. We use here the abstract framework developed by R. E. Bank and C. C. Douglas [B.D.] that fits exactly the numerical conclusion of [R.P.] concerning the optimal choice of the smoothing operations.

To our knowledge, the numerical analysis of the convergence for the multigrid algorithms used in spectral type techniques is somehow empty. The main reason is certainly that the nice analysis that can be done requires a variational framework and the awareness that the spectral methods are, exactly or very close to, variational approach is not so old. The other reason, perhaps, is that the previous multigrid techniques applied to spectral type methods [Z.W.H.1,2] used a finite difference preconditioner as a smoother and a Chebyshev framework. The convergence, in this case, is not so brillant as in the present approach and a priori more related to the good properties of preconditioner of the finite difference operator. Besides the variational formulation involves a nonsymmetric form that makes the analysis much more involved.

The paper is organized as follows; in section II, we recall the theory of [B.D.] in a form adapted to our analysis. In section III, we first explain on the test example of the Galerkin spectral approximation of the homogeneous Poisson problem the fundamental reasons of optimal properties of this multigrid method. The tools are based on Jackson inequality and some refined version of the approximation properties of the $L^2$-projection operator. In section IV, we generalize the analysis to the case of the spectral element approximation. We compare in each section the results obtained by the theory with the numerical results presented in [R.P.]. The last section V deals with the one domain multigrid technique when applied to a non constant second order problem.

The generalization of these results to multidimensional problems will be presented in a future paper.

## II. Position of the problem and abstract theorem.

### II.1 Generalities on variational multigrid techniques.

In this subsection, we first recall the theory developed by R. Bank and C. Douglas to analyze the convergence rate of the multigrid algorithm for solving the linear algebraic system that arises from the numerical approximation of elliptic partial differential equations. We present it in a version that we shall use afterwards. First of all, let $\mathcal{H}$ be a Hilbert space, a be a continuous, elliptic, symmetric, bilinear form and g be a continuous linear form, both defined over $\mathcal{H}$. The problem to be solved is : Find $u \in \mathcal{H}$ such that

(II.1) $\quad \forall\, v \in \mathcal{H}\ ,\ a(u,v) = g(v)\ .$

For the numerical resolution of this problem, we first introduce a sequence of finite dimensional subspaces $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots \subset \mathcal{M}_j$ of $\mathcal{H}$ ; then consider the problem : Find $u_j \in \mathcal{M}_j$ such that

$(II.2)_j \quad \forall\, v \in \mathcal{M}_j\ ,\ a(u_j,v) = g(v)\ .$

The basic idea for solving problem (II.2) with a multigrid algorithm consists in first defining a simple problem over the largest spaces $\mathcal{M}_j$ and solving it, then correcting the residual derived from the solution of this simpler problem when plugged into problem $(II.2)_j$ by solving problem $(II.2)_k$ for lower values of $k < j$. The first step is the most important one and relies on the good choice of continuous, elliptic, symmetric, bilinear form b, called smoother, that represents a in some sense and is easier to invert. Let us suppose that we have only two grids, the coarse one (i.e., $\mathcal{M}_1$ ) and the fine one (i.e., $\mathcal{M}_2$ , and $j = 2$). The two-grid procedure consists in

1- m/2 steps of smoothing where we solve m/2 times a problem like the following one : Find $\mathcal{S}\varphi$ in $\mathcal{M}_2$ such that

(II.3) $\quad \forall\, v \in \mathcal{M}_2\ ,\ b(\mathcal{S}\varphi-\varphi,v) = g(v) - a(\varphi,v)\ ,$

2- one step of coarse grid correction where we solve only once a problem like the following one: Find $\overline{\varphi}$ in $\mathcal{M}_1$ such that

(II.4) $\quad \forall\, v \in \mathcal{M}_1\ ,\ a(\overline{\varphi},v) = g(v) - a(\varphi,v)\ ,$

and define $\mathcal{C}\varphi = \varphi + \overline{\varphi}$.

3- m/2 steps of smoothing as in (II.3).

We consider here just two grid levels, the only reason is for sake of simplicity of the notations, but as in [B.D.], we could consider the whole W-cycles based on more than two levels. If

the initial guess for the exact solution u is $u^0$ , after one V-cycle like the one described previously by the three steps 1- 2- 3-, the resulting solution is $u^1$ and can be expressed like a function of $u^0$ as follows

(II.5) $\quad u^1 = \mathcal{S}^{m/2}\mathcal{C}\mathcal{S}^{m/2}(u^0)$ ,

so that, after the $r^{th}$ V-cycles, the solution is

(II.6) $\quad u^r = [\mathcal{S}^{m/2}\mathcal{C}\mathcal{S}^{m/2}]^r(u^0)$ .

Moreover, it is very simple to note that if u is the exact solution, then

$$\mathcal{C}u = u \quad \text{and} \quad \mathcal{S}u = u \ ,$$

so that, if $e^r$ denotes the error after the $r^{th}$ V-cycle, we derive from (II.6) that

(II.7) $\quad e^r = [\mathcal{S}^{m/2}\mathcal{C}\mathcal{S}^{m/2}]^r(e^0)$ .

Note that the equations (II.3) (II.4) define affine operators $\mathcal{C}$ and $\mathcal{S}$ but in (II.7) we can consider these operators as linear ones since they operate on differences $e^0$ and $e^r$; from now on, we shall consider these operator as linear ones while keeping the same notation.

As noted in [B.D.], of importance is the analysis of the spectrum of the following eigenvalue problem : Find $\Psi$ in $\mathcal{M}_2$ and $\lambda$ in $\mathbb{R}^+$ such that

(II.8) $\quad \forall\, v \in \mathcal{M}_2$ , $a(\Psi, v) = \lambda\, b(\Psi, v)$ ,

where b has been scaled so that the maximum eigenvalue $\lambda_{max}$ is equal to 1. Let us order the eigenvalues in increasing order $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_p = \lambda_{max} = 1$, (where P is the dimension of $\mathcal{M}_2$ ) and choose relative eigenfunctions $\Psi_1$ , $\Psi_2$ ,...,$\Psi_p$ . Of equal importance in the analysis is also the compatibility between the coarse space $\mathcal{M}_1$ and the space spanned by the first eigenvectors .

More precisely, under the following hypothesis

H the space $\mathcal{M}_1$ coincides with span $\{\Psi_1, \Psi_2, .., \Psi_p\}$, (where p is the dimension of $\mathcal{M}_1$ ) ;

then one can prove the following theorem

Theorem II.1 : *Under the hypothesis* H *the error after the first* V-cycle *verifies*

$$a(e^1, e^1) \leqslant (1-\lambda_{p+1})^{2m}\, a(e^0, e^0).$$

Proof : First of all, let us recall that the eigenfunctions $\Psi_n$ , n = 1,...,P, form a basis of $\mathcal{M}_2$ that

is orthogonal for both the forms a and b. Let us span $e^0$ in this basis; we get

$$e^0 = \sum_{n=1}^{N} \hat{e}_n^0 \Psi_n \quad ;$$

then, due to (II.3) and (II.8), we derive that $\mathcal{S}^{m/2}(e^0)$ satisfies

$$(II.9) \qquad \mathcal{S}^{m/2}(e^0) = \sum_{n=1}^{N} (1-\lambda_n)^{m/2} \hat{e}_n^0 \Psi_n \quad .$$

From hypothesis H, we then derive that the operator $\mathcal{C}$ truncates the previous spectrum so that

$$\mathcal{C}\mathcal{S}^{m/2}(e^0) = \sum_{n=p+1}^{N} (1-\lambda_n)^{m/2} \hat{e}_n^0 \Psi_n \quad ;$$

then the smoothing procedure diminishes once more the spectrum of the error as follows

$$e^1 = \mathcal{S}^{m/2}\mathcal{C}\mathcal{S}^{m/2}(e^0) = \sum_{n=p+1}^{N} (1-\lambda_n)^{m} \hat{e}_n^0 \Psi_n \quad .$$

We deduce now from the orthogonality of the $\Psi_n$ that

$$a(e^1,e^1) = \sum_{n=p+1}^{N} (1-\lambda_n)^{2m} (\hat{e}_n^0)^2 a(\Psi_n,\Psi_n) \leqslant (1-\lambda_{p+1})^{2m} \sum_{n=p+1}^{N} (\hat{e}_n^0)^2 a(\Psi_n,\Psi_n)$$
$$\leqslant (1-\lambda_{p+1})^{2m} \sum_{n=1}^{N} (\hat{e}_n^0)^2 a(\Psi_n,\Psi_n) = (1-\lambda_{p+1})^{2m} a(e^0,e^0) \quad .$$

Remark II.1 : Note that the previous theorem is very simple and is a trivial extension of the analysis of the multigrid procedure in the Fourier space. Note also that if the space $\mathcal{M}_1$ is not so well chosen, for instance if it coincides with span $\{\Psi_{N-p+1}, \Psi_{N-p+2}, .., \Psi_N\}$, then the multigrid procedure would not converge rapidly since, after the first V-cycle, we obtain

$$\mathcal{S}^{m/2}\mathcal{C}\mathcal{S}^{m/2}(e^0) = \sum_{n=1}^{N-p} (1-\lambda_n)^{m} \hat{e}_n^0 \Psi_n \quad ,$$

and the error remains important since $\lambda_1$ can be very small. In fact, the method has exactly the same properties as the plain Jacobi algorithm.

It turns out from the previous analysis that the multigrid procedure, when applied under hypothesis H, is degenerated since only one V-cycle is needed and m is the only important factor of convergence. So in the non trivial applications where the hypothesis H is not verified, we have to measure the actual situation between the hypothesis H and the situation explained in remark II.1. This can be explained as follow : the "rough" eigenmodes ( [B.D.] denotes this way the $\Psi_n$ with $\lambda_n$ close to 1) are damped during the smoothing procedure ( their components are multiplied by a factor $(1-\lambda_n)$ ) while the "smooth" modes ( corresponding to small $\lambda_n$) remain almost constant. Under the hypothesis H these are completely erased during the correction procedure, but if we are not in this optimal situation, they can only be damped also during this step.

The general analysis proposed in [B.D] allows for measuring the position with respect to both hypotheses. Let us recall the basis of their proof since we shall extend it in section V. To this purpose, they introduce the various norms, defined for any real index $\theta$, as follows

(II.10) $\quad \forall\, \varphi \in \mathcal{M}_2\ ,\ |||\, \varphi \,|||_\theta = [\sum_{n=1}^{N} \lambda_n^\theta \hat{\varphi}_n^2\ ]^{1/2}$ .

They introduce also the function $f$

(II.11) $\quad f(a,b) = a^a\, b^b\, (a+b)^{-(a+b)} = \sup_{x \in [0,1]} (1-x)^a\, x^b$ .

Then we derive that

(II.12) $\quad \forall\, \varphi \in \mathcal{M}_2\ ,\ \forall \tau,\ 0 \leqslant \tau \leqslant \theta\ ,\ |||\, \mathcal{S}^{m/2} \varphi \,|||_\theta = [\sum_{n=1}^{N} (1-\lambda_n)^m\, \lambda_n^\theta\, \hat{\varphi}_n^2\ ]^{1/2}$

$$\leqslant f(m/2,(\theta-\tau)/2)\ |||\, \varphi \,|||_\tau\ ;$$

let us write now

$$|||\, \mathcal{C}\mathcal{S}^{m/2} \varphi \,|||_1^2 = a(\, \mathcal{C}\mathcal{S}^{m/2}\varphi,\, \mathcal{C}\mathcal{S}^{m/2}\varphi) = a(\, \mathcal{C}\mathcal{S}^{m/2}\varphi,\, \mathcal{C}\mathcal{S}^{m/2}\varphi - \bar{\varphi}) = a(\, \mathcal{C}\mathcal{S}^{m/2}\varphi,\, \mathcal{S}^{m/2}\varphi)$$

$$\leqslant |||\, \mathcal{C}\mathcal{S}^{m/2}\varphi \,|||_{1-\beta}\ |||\, \mathcal{S}^{m/2}\varphi \,|||_{1+\beta}$$

$$\leqslant [\sup_{\psi \in \mathcal{M}_2} |||\, \mathcal{C}\psi \,|||_{1-\beta} / |||\, \mathcal{C}\psi \,|||_1\ ]\ |||\, \mathcal{C}\mathcal{S}^{m/2}\varphi \,|||_1\ |||\, \mathcal{S}^{m/2}\varphi \,|||_{1+\beta}\ ,$$

so that we derive

(II.13) $\quad |||\, \mathcal{C}\mathcal{S}^{m/2}\varphi \,|||_1 \leqslant [\sup_{\psi \in \mathcal{M}_2} |||\, \mathcal{C}\psi \,|||_{1-\beta} / |||\, \mathcal{C}\psi \,|||_1\ ]\ |||\, \mathcal{S}^{m/2}\varphi \,|||_{1+\beta}$ ,

which is valid for any $\beta > 0$; we deduce from (II.12) that

$$|||\, \mathcal{S}^{m/2}\mathcal{C}\mathcal{S}^{m/2}\varphi \,|||_1 \leqslant f(m/2,\beta/2)\ |||\, \mathcal{C}\mathcal{S}^{m/2}\varphi \,|||_{1-\beta}$$

$$\leqslant f(m/2,\beta/2)\ |||\, \mathcal{C}\mathcal{S}^{m/2}\varphi \,|||_1\ [\sup_{\psi \in \mathcal{M}_2} |||\, \mathcal{C}\psi \,|||_{1-\beta} / |||\, \mathcal{C}\psi \,|||_1\ ]$$

$$\leqslant f(m/2,\beta/2)\ [\sup_{\psi \in \mathcal{M}_2} |||\, \mathcal{C}\psi \,|||_{1-\beta} / |||\, \mathcal{C}\psi \,|||_1\ ]^2\ |||\, \mathcal{S}^{m/2}\varphi \,|||_{1+\beta}$$

$$\leqslant f(m,\beta)\ [\sup_{\psi \in \mathcal{M}_2} |||\, \mathcal{C}\psi \,|||_{1-\beta} / |||\, \mathcal{C}\psi \,|||_1\ ]^2\ |||\, \varphi \,|||_1\ .$$

Defining $\mathcal{M}_1^{\perp}$ as follows

$$\mathcal{M}_1^{\perp} = \{\, \varphi \in \mathcal{M}_2\ ,\ \text{such that } \forall\, \psi \in \mathcal{M}_1\ ,\ a(\varphi,\psi) = 0\ \}\ ,$$

( that coincides with the range of $\mathcal{C}$ ) and by minimizing the right-hand side over $\beta$, they state the following

<u>Theorem II.1</u> : *Assume there exists a constant* $\kappa \geqslant 1$ *and* $\alpha > 0$ *such that for any* $\varphi \in \mathcal{M}_1^{\perp}$

$$|||\, \varphi \,|||_{1-\alpha} \leqslant \kappa^{\alpha/2}\ |||\, \varphi \,|||_1\ ,$$

*then*

(II.14) $\quad a(e^1,e^1) \leqslant \gamma\, a(e^0,e^0)$ ,

*where*

(II.15) $\gamma = [(\kappa-1)/\kappa]^{2m}$ if $m \leqslant \alpha(\kappa-1)$

$\gamma = (\kappa^{\alpha} f(m,\alpha))^2$ if $m > \alpha(\kappa-1)$ .

## II.2 Formulation of the spectral element discretization.

Let us turn now to the position of the problem. We consider here the simple test problem over the interval $\Lambda = ]-1,1[$ : Find u such that

(II.16) $-u_{xx} = f$ , over $\Lambda$

provided with homogeneous Dirichlet boundary conditions

(II.17) $u(-1) = u(1) = 0.$

where f is a given force. This problem is very simple, but it allows a statement of the basic features of the multigrid algorithm and an understanding of why the method works.

The spectral element method for approximating the solution of (II.16) consists in discretizing the space of acceptable functions by a subspace of piecewise polynomials. More precisely, given a pair $h = (K,N)$, we first break the interval $\Lambda$ into K disjoint subintervals of comparable sizes

$\Lambda = \bigcup_{k=1}^{K} \Lambda_k$ , $\Lambda_k = ] a_k , a_k + b_k [$ ;

then, we choose for space of approximate functions a subspace $X_h^N$ of $H_0^1(\Lambda)$ consisting of all piecewise polynomials of degree $\leqslant N$,

(II.18a) $X_h^N = Y_h^N \cap H_0^1(\Lambda)$ ,

where

(II.18b) $Y_h^N = \{ \varphi \text{ such that } \varphi_{|\Lambda_k} \in \mathbb{P}_N(\Lambda_k) \}$

and $\mathbb{P}_N(\Lambda_k)$ denotes the space of all polynomials of degree $\leqslant N$ on $\Lambda_k$ . We remind that contrarily to the finite element approximation, the convergence is achieved by increasing the degree of the polynomials N and not refining the mesh.

The discrete problem starts from the variational formulation of problem (II.16)(II.17) that is : Find u in $H_0^1(\Lambda)$ such that

(II.19) $\forall v \in H_0^1(\Lambda)$, $a(u,v) = (f,v)$ ,

where $(.,.)$ denotes the $L^2(\Lambda)$-scalar product and $a(.,.)$ denotes the following bilinear form defined over $H_0^1(\Lambda)$ as follows

$\forall\ \varphi,\psi \in H_0^1(\Lambda)\ ,\ \ a(\varphi,\psi) = (\varphi_x,\psi_x)\ .$

Then we construct the numerical scheme by discretizing with Gauss-Lobatto quadrature formulas the various integrals present in (II.19) and restricting the space of test functions to $X_h^N$.
This results in the following problem : Find $u_h \in X_h^N$ , such that

(II.20) $\quad \forall\ v_h \in X_h^N\ ,\ \ a_{h,GL}^N(u_h,v_h) = (f,v_h)_{h,GL}^N\ ,$

where the discrete forms are defined as follows

$$\forall\ \varphi,\psi \in Y_h^N\ ,\ \ (\varphi,\psi)_{h,GL}^N = \sum_{k=1}^K b_k/2 \sum_{n=0}^N \varrho_n^N\, \varphi(\xi_{n,k})\, \psi(\xi_{n,k})\ ,$$

$$\forall\ \varphi,\psi \in X_h^N\ ,\ \ a_{h,GL}^N(\varphi,\psi) = (\varphi_x\ ,\ \psi_x)_{h,GL}^N\ .$$

Here, the $\varrho_n^N$ , and the $\xi_n^N$ are the weights and nodes of the Gauss Lobatto-Legendre formula with $N+1$ points and the collocation points $\xi_{n,k}$ are defined by $\xi_{n,k} = a_k + (\xi_n^N + 1)b_k/2$ . W recall here that the integration formula is exact on $\mathbb{P}_{2N-1}$ so that, contrary to the appearances, $a_{h,GL}^N$ does not depend on h nor N since $a_{h,GL}^N = a$ ( see e.g. [M.P.]).

The algebraic system that has to be solved is derived by choosing the values of the unknown function $u_h$ on the collocation points and representing $u_h$ in the basis of the interpolant basis $h_{k,n}$ defined by : $h_{k,n}$ is the only element of $Y_h^N$ such that

(II.21) $\quad \forall\ \xi_{\ell,m}\ ,\ h_{k,n}(\xi_{\ell,m}) = \delta_{k\ell}\,\delta_{nm}\ .$

The matrix system that has to be solved can be written as follows: Find $\mathbf{u} = (u_h^{k,n})$ such that

(II.22) $\quad \mathbf{A}\,\mathbf{u} = \mathbf{g}\ ,$

where $\mathbf{A}$ is the stiffness matrix with entries equal to

(II.23) $\quad (2/b_k)\sum'^{K}_{k=1} \sum_{n=0}^N \varrho_{n,k}\,[dh_{i,\ell}/dx\ dh_{j,m}/dx]\,(\xi_{n,k})\ ,$

with $\varrho_{n,k} = \varrho_n\, b_k/2$ and $\sum'$ denotes the direct "stiffness summation" , while $\mathbf{g}$ is related to the forcing term. We refer to [R.P.] so as to [M.P.] for more details on the derivation of this matrix.

This numerical method is proved to work very well in a great number of interesting problems that include, for instance, the full Navier-Stokes problem (see for instance [M.P.], [M.P.R.1], [M.P.R.2] or [R]) and is numericaly competitive and implementable on a parallel medium grain paradigm (see for instance [F.R.D.P.]).

Let us turn now to the multigrid algorithm for solving iteratively problem (II.22). As

explained in [R.P.] , nested spaces are related to spaces of polynomials with lower degree $X_h^M$ (say $M = N/2$) and the smoother is simply the Jacobi preconditioner B that is proportional to the diagonal part of the matrix A and normalized in such a way that the highest eigenvalue of $B^{-1}A$ is 1. We shall analyze in the two following sections the properties of this preconditioner and explain why the numerical method works so well as presented in [R.P.] .

In this paper, we shall use two grids only though the analysis can be performed with no extra difficulty than the one of comprehension, and we shall use

$$\mathcal{M}_1 = X_h^{N/2} \quad \text{and} \quad \mathcal{M}_2 = X_h^N \ .$$

The problem enters in the general theory of [B.D.] since the numerical problem involves bilinear forms and matrices that do not depend on h.

## III. Analysis of the convergence of the multigrid algorithm in the case of one element.

As announced in the title, we shall assume for the moment that the discretization is applied on only one domain to problem (II.16). For sake of simplicity, we shall drop out the second index that corresponds to the subelement characterization in the notations. The situation is simple here as soon as we have explained the properties of the diagonal matrix, but this simple example permits enhancement of the main features that allow for a rapid convergence of the algorithm. The problem can be written as a pure collocation scheme as already noted in [M.P.]. Indeed, by taking $v_h = h_n$ for $n = 1,..,N-1$, in (II.20) and using twice the exactness of the Gauss-Lobatto formula, we derive

$$a^N_{h,GL}(u_h,h_n) = a(u_h,h_n) = - \int_\Lambda u_h''(x)\, h_n(x)\, dx = - ( u_h'' , h_n)_{h,GL} = - u_h''(\xi_n)\, \varrho_n ;$$

besides we note that

$$(f, h_n)^N_{h,GL} = f(\xi_n)\, \varrho_n ,$$

so that the problem actually verified by $u_h$ is : Find $u_h$ in $X^N_h$ such that

$$\forall\, n,\ n=1,..,N-1 \ , \quad - u_h''(\xi_n) = f(\xi_n) \ .$$

This consists in a pure collocation procedure to solve the initial problem.

In order to analyze the multigrid algorithm, let us first compute exactly the diagonal part of the stiffness matrix; that is here simply to the matrix with entries equal to

$$A_{ij} = \sum^N_{n=0} \varrho_n\, h_i'(\xi^N_n)\, h_j'(\xi^N_n) \quad , \text{ for } i,j = 1,...,N-1 \ ,$$

$$A_{0j} = A_{Nj} = 0.$$

Due to the exactness of the Gauss Lobatto formula, we easily obtain that, for $i = 1,...,N-1$,

$$A_{ii} = \int_\Lambda [h_i'(x)]^2\, dx = - \int_\Lambda h_i''(x)\, h_i(x)\, dx \ ,$$

and here again, by using the exactness of the Gauss Lobatto formula, we derive from (II.21)

$$A_{ii} = - \sum^N_{n=0} \varrho_n\, h_i''(\xi^N_n)\, h_i(\xi^N_n) = -\varrho_i\, h_i''(\xi^N_i) \ .$$

As already derived in [GHO, formula (7.4)]

Lemma III.1 : *The diagonal of the stiffness matrix verifies*

(III.1) $\quad A_{ii} = \varrho_i\, N\,(N+1)/[3(1-(\xi^N_i)^2)]$ .

Proof : Let us drop out in this proof, the superscript N. First of all, we note that, from the definition (II.21) of $h_i$, we have

$$h_i(x) = \frac{\prod_{\substack{j=1 \\ j\neq i}}^{N-1} (x-\xi_j)(x^2-1)}{\prod_{\substack{j=1 \\ j\neq i}}^{N-1} (\xi_i-\xi_j)(\xi_i^2-1)} = \alpha_i \prod_{\substack{j=1 \\ j\neq i}}^{N-1} (x-\xi_j)\,(1-x^2)$$

where $\alpha_i$ is a non zero constant. It is well known (see, e.g [D.R.]) that the internal nodes of the Gauss Lobatto fomula verify

(III.2) $\quad L_N'(x) = c_N \prod_{j=1}^{N-1} (x-\xi_j)$ ,

where $c_N$ is a non zero constant. Hence, we can write

$$(x-\xi_i)\, h_i(x) = c_N^{-1}\, \alpha_i\, (1-x^2)\, L_N'(x) = \tilde{\alpha}_i\, (1-x^2)\, L_N'(x),$$

and after taking the derivative of both sides, we obtain

$$(x-\xi_i)\, h_i'(x) + h_i(x) = \tilde{\alpha}_i\, (d/dx)[(1-x^2)\, L_N'(x)] \ .$$

Let us recall now the eigenfunctions property verified by the Legendre polynomials (see, e.g. [D.R.])

(III.3) $\quad (d/dx)[(1-x^2)\, L_N'(x)] = -(N)(N+1)\, L_N(x)$ ;

we derive that

(III.4) $\quad (x-\xi_i)\, h_i'(x) + h_i(x) = -\,\tilde{\alpha}_i\, N\,(N+1)\, L_N(x)$ ;

plugging now $x = \xi_i$ in this equation yields

(III.5) $\quad 1 = -\,\tilde{\alpha}_i\, N\,(N+1)\, L_N(\xi_i)$ .

Besides, by taking the derivative of (III.4), we obtain

(III.6) $\quad (x-\xi_i)\, h_i''(x) + 2\, h_i'(x) = -\,\tilde{\alpha}_i\, N\,(N+1)\, L_N'(x)$ ,

plugging also here $x = \xi_i$ in this equality and using (III.2), we get

(III.7) $\quad h_i'(\xi_i) = 0$ .

Let us multiply now (III.6) by $(1-x^2)$ and take the derivative of the resulting equation, we deduce

$$(x-\xi_i)\, (d/dx)[(1-x^2)\, h_i''(x)] + 3(1-x^2)\, h_i''(x) - 4x h_i'(x) = \tilde{\alpha}_i\, N^2\, (N+1)^2\, L_N(x) \ .$$

Finally, plugging one more time $x = \xi_i$ , we derive from (III.5)

$$3\,(1-\xi_i^2)\, h_i''(\xi_i) = -\,N\,(N+1) \ ,$$

and the lemma follows from the value of $A_{ii}$ .

<u>Remark III.1</u> : Note that, as a consequence of (III.7), we have proved here that the Lagrangian at point $\xi_i$ has its maximum at $\xi_i$ .

Associated with the matrix A is the bilinear form a, and in the same way, associated with B, the normalized diagonal of A can be defined a bilinear form $b^N_{h,GL}$ .This will be the smoother of the multigrid algorithm. Lemma III.1 proves that the smoother we have introduced is proportional to the bilinear form $\tilde{b}^N_{h,GL}$ defined for any $\varphi$ and $\psi$ in $X^N_h$ (which here is simply $\mathbb{P}_N(\Lambda) \cap H^1_0(\Lambda)$)

(III.8) $\quad \tilde{b}^N_{h,GL}(\varphi,\psi) = \sum^N_{i=0} \varrho_i \varphi(\xi_i) \psi(\xi_i) (1-\xi_i^2)^{-1}$ .

It is interesting to note that, due to the exactness of the Gauss-Lobatto formula, we can rewrite $\tilde{b}^N_{h,GL}$ in a continuous way since for any $\varphi$ and $\psi$ in $X^N_h$ , the function $[\varphi\ \psi\ (1-x^2)^{-1}]$ is still a polynomial and belongs to $\mathbb{P}_{2N-2}(\Lambda)$, so that we have

(III.9) $\quad \tilde{b}^N_{h,GL}(\varphi,\psi) = \tilde{b}(\varphi,\psi) = \int_\Lambda \varphi(x) \psi(x) (1-x^2)^{-1} dx$ .

Let us now analyze the eigenvalue problem (II.8) or more precisely the eigenvalue problem associated to $\tilde{b}$. The situation is here very simple since we have an exact expression for the solution to this problem

Lemma III.2 : *Let us define for any integer* $n$ , $1 \leqslant n \leqslant N-1$

(III.10) $\quad \Psi_n(x) = (1-x^2) L'_n(x)$ ;

*then we have*

(III.11) $\quad \forall\, v \in X^N_h$ , $a(\Psi_n,v) = n(n+1)\, \tilde{b}(\Psi_n,v)$ .

Proof : Let us first remind the following standard formula on the Legendre polynomial (see, e.g. [D.R., Chap 2, §7] )

(III.12) $\quad \forall\, n \in \mathbb{N}$ , $\quad ((1-x^2) L'_n(x))' + n(n+1) L_n(x) = 0.$

Let us compute, for any v in $X^N_h$ ,

$$a(\Psi_n,v) = \int_\Lambda \Psi'_n(x)\, v'(x)\, dx = \int_\Lambda ((1-x^2) L'_n(x))'\, v'(x)\, dx\ ;$$

using (III.12) and integrating by parts, we obtain

$$\begin{aligned} a(\Psi_n,v) &= -\, n(n+1) \int_\Lambda L_n(x)\, v'(x)\, dx = n(n+1) \int_\Lambda L'_n(x)\, v(x)\, dx \\ &= n(n+1) \int_\Lambda (1-x^2) L'_n(x)\, v(x)\, (1-x^2)^{-1} dx = n(n+1)\, \tilde{b}(\Psi_n, v); \end{aligned}$$

this ends the proof of Lemma III.2 .

It is important to note that in this simple example the eigenvalues are well known and, moreover, that the first M of them span exactly the space $\mathbb{P}_{M+1}(\Lambda) \cap H^1_0(\Lambda)$. As a first consequence, we can state that the normalized form that will be used as a smoother is defined as

follows

(III.13) $\forall\ \varphi, \psi \in X_h^N\ ,\ b(\varphi,\psi) = N(N-1)\,\tilde{b}(\varphi,\psi) = N(N-1)\int_\Lambda \varphi(x)\,\psi(x)/(1-x^2)\,dx\ .$

We demonstrate the case where the simple hypothesis H of section II is valid, and we can state now

Theorem III.1 : *Let* $u^0$ *denote the initial guess in the multigrid algorithm applied to problem* (II.20) *in the case of one element, and* $u^1$ *denotes the solution obtained in the after* m *smoothing and one correction. This solution converges to the exact solution* u *as follows,*

$$a(u-u^1, u-u^1) = [1 - (N+2)/4(N-1)]^{2m}\, a(u-u^0, u-u^0)\ .$$

Proof : This is a simple corollary of Theorem II.1 and lemma III.2 since the first $(N/2)-1$ eigenvectors $\Psi_n$ span the coarse space $\mathcal{M}_1 = X_h^{N/2}$ ; it follows that the eigenvalues of problem (II.8) are $n(n+1)/(N-1)N$.

Remark III.2 : Note here that the correction on the coarse grid needs only be done once and that there is no optimal choice for the number of smoothings since the convergence is proportional to this number. This is actually in accordance with the numerical simulation of [R.P.] as appears in table 1 of that paper.

Remark III.3 : Let us point out the fundamental reasons that give this rapidity to the algorithm. They are hidden here due to the simplicity of the eigenvalue problem. First of all, even if this is not of major importance, the matrix B is a good preconditioner of the matrix A. Indeed, the condition number of $B^{-1}A$ is order $N^2$ as opposed to the condition number of A which is order $N^3$. This will be also the case for other problems and has already been noticed by [H.] in a different context for an application to conjugate gradient algorithms. This can be viewed as an inverse inequality or a Jackson type one since the weighted $L^2$-type norm associated to the bilinear form $\tilde{b}$ is compared to the $H^1(\Lambda)$-norm as follows

(III.14) $\forall\ \varphi \in \mathbb{P}_N(\Lambda) \cap H_0^1(\Lambda)\ ,\ \|\varphi\|_{1,\Lambda}^2 \leq N(N-1)\,\tilde{b}(\varphi,\varphi)\ .$

Secondly, the other property that is very important in the multigrid algorithm is that the factor $\kappa$ as defined in Theorem II.1 is also bounded by two constants independent of N. Indeed, we can state

$$\forall\ \varphi \in X_h^N \text{ such that } a_{h,GL}^N(\varphi, v) = 0 \text{ for any } v \text{ in } X_h^{N/2}$$

we have

(III.15) $b(\varphi,\varphi) \leq \kappa^{1/2}\, a(\varphi,\varphi)\ ,$

where the constant $\kappa$ is bounded by $4(N-1)/(N+2)$ as can be derived from lemma III.2 .

Let us generalize now the results obtained we have obtained in this very simple situation to the case of a multielement discretization.

## IV. Analysis of the convergence of the multigrid algorithm in the case of several elements.

We begin here also by analyzing the properties of the diagonal of the stiffness matrix A. We immediately note that there are two kinds of diagonal elements in this matrix : those that correspond to internal points, i.e. that involve the scalar product

(IV.1) $A_{i,\ell} = \sum_{k=1}^{K} \sum_{n=0}^{N} \varrho_{k,n} [dh_{i,\ell}/dx \; dh_{i,\ell}/dx](\xi_{n,k})$

with $i = 1,..,N-1$, and those that correspond to interface points, i.e., that involve the scalar product (IV.1) for $i = 0$ or N. The first kind of diagonal elements are the same as those involved in the previous section. Indeed, the corresponding Lagrangian interpolants vanish at the interfaces and also on any subinterval that does not contain the point $\xi_{i,\ell}$ ; therefore, from Lemma III.1, we can state that

$$\forall\, i,\ i = 1,..,N-1\ ,\ \forall\, \ell\ ,\ell = 1,..,K\ ,\ A_{i,\ell} = (2/b_\ell)\, \varrho_i\, N(N+1)/[3\,(1-(\xi_i)^2)]\ ,$$

or again, thanks to a simple change of variable

(IV.2) $$\forall\, i,\ i = 1,..,N-1\ ,\ \forall\, \ell\ ,\ell = 1,..,K\ ,\ A_{i,\ell} = \varrho_{i,\ell}\, N(N+1)/[3(\xi_{i,\ell} - a_\ell)(a_{\ell+1} - \xi_{i,\ell})]\ .$$

For the interface terms we have

Lemma IV.1 : *For* $i = 0$ *and any* $\ell = 2,..,K$ *we have*

(IV.3) $$A_{i,\ell} = [(b_{\ell-1})^{-1} + (b_\ell)^{-1}]\,(N^2 + N + 1)/3\ .$$

Proof : As already used, the exactness of the Gauss-Lobatto formula gives, for $i = 0$ and $\ell = 2,..,K$ ( or $i = N$ and $\ell = 1,..,N-1$, due to the direct stiffness summation)

(IV.4) $$A_{i,\ell} = \int_{\Lambda_{\ell-1}} [dh_{N,\ell-1}/dx]^2(\xi)\, d\xi + \int_{\Lambda_\ell} [dh_{0,\ell}/dx]^2(\xi)\, d\xi\ .$$

A simple change of variables and the use of the symmetry of $h_N$ and $h_0$ yields

(IV.5) $$A_{i,\ell} = [(2/b_{\ell-1}) + (2/b_\ell)] \int_\Lambda [dh_N/dx]^2(\xi)\, d\xi\ .$$

Let us compute the integral on the right-hand side of this equation. First we have

$$h_N(x) = \frac{(x+1)\prod_{i=1}^{N-1}(x-\xi_i)}{2\prod_{i=1}^{N-1}(1-\xi_i)}\ .$$

From (III.13) we then get that

$$h_N(x) = (x+1)L_N'(x)/2L_N'(1)\ ,$$

so that, after integration by parts, we obtain

$$\int_\Lambda [dh_N/dx]^2(x)\,dx = -(2\,L_N'(1))^{-2}\,[\int_\Lambda (x+1)\,L_N'(x)\,(d^2/dx^2)((x+1)\,L_N'(x))\,dx - 2\,L_N'(1)\,(d/dx)((x+1)\,L_N'(x))(1)\,]\ ;$$

here again, the use of the exactness of the Gauss-Lobatto formula to compute the integral on the right-hand side yields

(IV.6) $$\int_\Lambda [dh_N/dx]^2(x)\,dx = -(2\,L_N'(1))^{-1}\,[(d^2/dx^2)((x+1)\,L_N'(x))(1)\,\varrho_N - (d/dx)((x+1)\,L_N'(x))(1)]\ .$$

It is an easy matter to note that

$$(d/dx)((x+1)\,L_N'(x)) = (x+1)\,L_N''(x) + L_N'(x)\ ,$$

$$(d^2/dx^2)((x+1)\,L_N'(x)) = (x+1)\,L_N'''(x) + 2\,L_N''(x)\ ,$$

from (III.12) writen in the form

$$(1-x^2)\,L_N''(x) - 2xL_N'(x) + N(N+1)\,L_N(x) = 0\ ;$$

we derive that

$$L_N'(1) = N(N+1)/2\ ,$$

$$L_N''(1) = (N-1)N(N+1)(N+2)/8\ ,$$

$$L_N'''(1) = (N-2)(N-1)N(N+1)(N+2)(N+3)/48\ ;$$

this gives

$$(d/dx)((x+1)\,L_N'(x))(1) = N^2(N+1)^2/4\ ,$$

$$(d^2/dx^2)((x+1)\,L_N'(x))(1) = (N-1)N^2(N+1)^2(N+2)/24\ .$$

Plugging this in (IV.6) and using the relation (see [D.R.])

(IV.7) $$\varrho_N = 2/(N+1)N\ ,$$

we derive

(IV.8) $$\int_\Lambda [dh_N/dx]^2(x)\,dx = (N^2+N+1)/6.$$

The lemma follows then from (IV.5).

From (IV.2) (IV.3) and (IV.7), we derive that the bilinear form $b^N_{h,GL}$ that defines the smoother and is associated to the normalized diagonal part of A is proportional to the bilinear form $\tilde{b}^N_{h,GL}$ defined for any $\varphi$ and $\psi$ in $X^N_h$ as follows (remind (III.8))

(IV.9) $$\tilde{b}^N_{h,GL}(\varphi,\psi) = \sum_{\ell=1}^K \tilde{b}_\ell(\varphi,\psi)\ ,$$

where

$$\tilde{b}_\ell(\varphi,\psi) = \{\sum_{i=1}^{N-1} \varrho_{i,\ell}\,\varphi(\xi_{i,\ell})\,\psi(\xi_{i,\ell})\,[(\xi_{i,\ell}-a_\ell)(a_{\ell+1}-\xi_{i,\ell})]^{-1}$$
$$+ \varrho_{0,\ell}\,\varphi(\xi_{0,\ell})\,\psi(\xi_{0,\ell})\,[(N^2+N+1)/(b_\ell)^2] + \varrho_{N,\ell}\,\varphi(\xi_{N,\ell})\,\psi(\xi_{N,\ell})\,[(N^2+N+1)/(b_\ell)^2]\}\,.$$

In the two next lemmas, we shall analyze the eigenvalue problem between a and $\tilde{b}^N_{h,GL}$ that will allow first to estimate the normalization factor between $b^N_{h,GL}$ and $\tilde{b}^N_{h,GL}$.

Lemma IV.2 : *For any* $\varphi$ *in* $X^N_h$ *, we have*

(IV.10) $$\int_\Lambda \varphi'^2(x)\,dx \leqslant (4N^3+2N^2+3N-1)N/3(N^2+N+1)\,\tilde{b}^N_{h,GL}(\varphi,\varphi)\,.$$

Proof : To any $\varphi$ in $X^N_h$ let us associate the element $\varphi_0$ defined as follows

(IV.11) $$\varphi_0 = \varphi - \sum_{\ell=2}^{K} \varphi(a_\ell)\,[h_{0,\ell} + h_{N,\ell-1}]\,,$$

so that $\varphi_0$ and $\varphi$ coincide on the interior collocation points. Using the inequality

$$(a+b)^2 \leqslant (1+\alpha^{-1})\,a^2 + (1+\alpha)\,b^2\,,$$

we deduce from (IV.11),

(IV.12) $$\int_{\Lambda_\ell} \varphi'^2(x)\,dx \leqslant (1+\alpha^{-1})\int_{\Lambda_\ell} \varphi_0'^2(x)\,dx + (1+\alpha)\int_{\Lambda_\ell} [\varphi(a_\ell)h'_{0,\ell} + \varphi(a_{\ell+1})h'_{N,\ell}]^2(x)\,dx.$$

It is an easy matter to note that the restriction of $\varphi_0$ to any $\Lambda_\ell$ belongs to $\mathbb{P}_N(\Lambda_\ell) \cap H^1_0(\Lambda_\ell)$ so that the lemma III.2 and a simple change of variables yields

$$\int_{\Lambda_\ell} \varphi_0'^2(x)\,dx \leqslant N(N-1)\sum_{i=1}^{N-1} \varrho_{i,\ell}\,\varphi_0^2(\xi_{i,\ell})[(\xi_{i,\ell}-a_\ell)(a_\ell+b_\ell-\xi_{i,\ell})]^{-1}$$
$$\leqslant N(N-1)\sum_{i=1}^{N-1} \varrho_{i,\ell}\,\varphi^2(\xi_{i,\ell})[(\xi_{i,\ell}-a_\ell)(a_\ell+b_\ell-\xi_{i,\ell})]^{-1}.$$

Besides, from (IV.8), we derive that

$$\int_{\Lambda_\ell} [h'_{0,\ell}]^2(x)\,dx = \int_{\Lambda_\ell} [h'_{N,\ell}]^2(x)\,dx = (N^2+N+1)/3b_\ell\,,$$

as following the same lines as in the proof of lemma IV.1 ; we get, for any $N \geqslant 2$

$$\int_{\Lambda_\ell} h'_{0,\ell}\,h'_{N,\ell}(x)\,dx \leqslant 1/3b_\ell\,.$$

From (IV.12) we then deduce

$$\int_{\Lambda_\ell} \varphi'^2(x)\,dx \leqslant (1+\alpha^{-1})N(N-1)\sum_{i=1}^{N-1} \varrho_{i,\ell}\,\varphi^2(\xi_{i,\ell})[(\xi_{i,\ell}-a_\ell)(a_\ell+b_\ell-\xi_{i,\ell})]^{-1}$$
$$+ (1+\alpha)[(N^2+N+2)/3b_\ell][|\varphi(a_\ell)|^2 + |\varphi(a_{\ell+1})|^2],$$

choosing now $\alpha = 3(N-1)(N^2+N+1)/(N+1)(N^2+N+2)$; it follows from (IV.7) and (IV.9) that

$$\int_\Lambda \varphi'^2(x)\,dx \leqslant (4N^3+2N^2+3N-1)N/3(N^2+N+1)\,\tilde{b}^N_{h,GL}(\varphi,\varphi)\,,$$

and the lemma follows.

Remark IV.1 : The estimate in the previous lemma provides a less precise characterization of the eigenvalue problem than the one we could get in lemma III.1, but we note that the highest

eigenvalue involves the same asymptotic order as in the previous section and this will be enough for our purpose. The important fact is that **the result is independent of K** and of the ratios between the various subinterval length $b_\ell$. As we shall see in what follows, this will result in a multigrid algorithm that will work as well in any case of number of subelements. Besides, note that the smallest eigenvalue of problem (II.8) scales like $K^{-2}$ (independently of N) such that the condition number of $B^{-1}A$ behaves like $(KN)^2$, in accordance with the finite element equivalent (when N is order 1) and proves that the conjugate gradient algorithm, when preconditioned by B, has a rate of convergence which behaves like $1 - c/KN$ and depends on both K and N!!! This is of importance when we compare the preconditioned conjugate gradient with the multigrid algorithm. We refer to the thesis of E. M. RONQUIST [R] for numerical evidences.

It is an easy matter to derive from lemma IV.2 that the normalized form $b^N_{h,GL}$ is defined from $\tilde{b}^N_{h,GL}$ by multiplication by a factor of order $(4/3)N^2$. The other property that is important for the analysis of the multigrid algorithm deals with the space $X_h^{N/2\perp}$ of those elements $\varphi$ of $X_h^N$ that verify

(IV.13) $\quad \forall\, \psi \in X_h^{N/2}\ ,\ a^N_{h,GL}(\varphi,\psi) = 0\ .$

Lemma IV.3 : *For any* $\varphi$ *in* $X_h^{N/2\perp}$ *we have*

(IV.14) $\quad b^N_{h,GL}(\varphi,\varphi) \leqslant 4(4N^3+2N^2+3N-1)/3(N^2+N+1)(N+2)\, a(\varphi,\varphi)\ .$

Proof : Let $\varphi$ belong to $X_h^{N/2\perp}$, and let us define for $\ell = 2,..,K$ the element $\psi_\ell$ of $X_h^{N/2}$ by

$$\psi_\ell \in X_h^1\ ,\ \forall\, k = 1,...,K+1,\ \ \psi_\ell(a_k) = \delta_{\ell k}\ .$$

It is an easy matter to compute that

$$\psi'_\ell(x) = \begin{cases} 1/b_{\ell-1}\ , & \text{for } x \text{ in } \Lambda_{\ell-1}\ , \\ -1/b_\ell\ , & \text{for } x \text{ in } \Lambda_\ell\ , \\ 0\ , & \text{for } x \text{ in } \Lambda_k\ ,\ k \neq \ell \text{ and } k \neq \ell-1\ . \end{cases}$$

Using this function in (IV.13), we derive that

$$\forall\, \ell = 2,..,K\ ,\ (1/b_{\ell-1})(\varphi(a_\ell) - \varphi(a_{\ell-1})) + (1/b_\ell)(\varphi(a_\ell) - \varphi(a_{\ell+1})) = 0\ ;$$

recalling now that $\varphi(a_0) = \varphi(a_{K+1}) = 0$, we deduce that in fact $\varphi$ vanishes at any interface so that $\varphi_{|\Lambda_\ell}$ is an element of $\mathbb{P}_N(\Lambda_\ell) \cap H^1_0(\Lambda_\ell)$. The use of (III.15) over each $\Lambda_\ell$ proves that

$$\tilde{b}^N_{h,GL}(\varphi,\varphi) = \sum_{\ell=1}^K \tilde{b}_\ell(\varphi,\varphi) \leqslant 4/N(N+2)\, a(\varphi,\varphi)\ .$$

and the lemma follows by recalling the normalization factor of $b^N_{h,GL}$.

As a simple consequence of the previous lemmas and (II.11)(II.14) we derive that the multigrid algorithm converges toward the numerical solution with a speed independent of both N and K, indeed, we have

Theorem IV.1 : *The multigrid algorithm based on the Jacobi preconditioner converges, and at any* V-cycle *with* m *smoothing as detailed in section* II *, the rate of convergence is given by*

$$\int_\Lambda e_1'^2(x)\,dx \leqslant \gamma \int_\Lambda e_0'^2(x)\,dx \quad ,$$

*where*

$$\gamma = [(\kappa-1)/\kappa]^{2m} \quad , \quad \text{for } m \leqslant (\kappa-1) \ ,$$

*and*

$$\gamma = (\kappa f(m,1))^2 \quad , \quad \text{for } m > (\kappa-1) \ ,$$

*with*

$$\kappa = 4(4N^3+2N^2+3N-1)/3(N^2+N+1)(N+2) = (16/3)\,(1+\mathcal{O}(N^{-1})) \quad .$$

Remark IV.2 : Let us note that the convergence rate that we have theoretically obtained is independent of K and the sizes of the subintervals and does not deteriorate when N increases; this is in accordance with what is numerically observed in part one of this paper [R.P.]. However, the rather rough estimate we gave for the highest eigenvalue in (IV.10) provides a rather too high estimate for the convergence parameter ( close to 0.81 when N is large enough ). Note also that the optimal choice of parameter $4 < m < 5$ is close to what can be observed numerically. As in the analysis of the multigrid algorithm when a finite element method is used, this optimal parameter is lower than the actual one. By using negative order of the norms defined from b and a, we could fit more closely to the experiments for this last result. This would be obtained, however, at the price of a much more technical proof and would not really be worthy since the main conclusion is the independence of the convergence rate with respect to the parameters of the discretization.

## V. Analysis of the convergence of the multigrid algorithm in the case of non constant coefficients.

The previous chapter was devoted to the analysis of the multigrid algorithm when applied to the very simple equation $-u_{xx} = f$. Of interest of course is to know that the same conclusions hold true in more complicated situations. We shall extend here the analysis to the case of the equation

(V.1) $-(\alpha u_x)_x = f$ ,

where $\alpha$ is a function of x such that there exists two constants $\alpha^-$ and $\alpha^+$ with

(V.2) $\forall\, x \in \Lambda\ ,\ 0 < \alpha^- \leqslant \alpha(x) \leqslant \alpha^+$ ,

and also such that $\alpha$ is in the Sobolev space $W^{2,\infty}(\Lambda)$. We shall assume here that the domain is not decomposed into subdomains, leaving this analysis to a forthcoming paper as well as the analysis of the multidimensional case. The analysis provided here is inspired from the reference [B.V.].

It is standard to note here that problem (V.1) can also be stated in a variational formulation like (II.19) with now a defined as follows

(V.3) $\forall\, \varphi,\psi \in H_0^1(\Lambda)\ ,\ a(\varphi,\psi) = \int_\Lambda \alpha(x)\, \varphi'(x)\, \psi'(x)\, dx$ .

It is rather well known also that the general spectral(element) discretization of the equation consists in first choosing a discretization parameter $n \in \mathbb{N}^*$, then : Find $u_n \in X^n = \mathbb{P}_n(\Lambda) \cap H_0^1(\Lambda)$ such that

(V.4) $\forall\, v_n \in X^n\ , a^n(u_n,v_n) = (f,v_n)_n$ ,

where the discrete form $a^n$ is defined here by

(V.5) $\forall\, \varphi, \psi \in X^n\ ,\ a^n(\varphi,\psi) = (\alpha\, \varphi_x\, ,\, \psi_x)_n$ .

The nested spaces for the multigrid algorithm are exactly the same as in the previous sections, i.e., $\mathcal{M}_1 = X^{N/2}$ and $\mathcal{M}_2 = X^N$ , and the strategy also based on the use of the Jacobi preconditioner as a smoother. The corresponding bilinear form $b^N$ is deduced from the following $\hat{b}^N$ after normalization of the maximum eigenvalue of problem (II.7): $\hat{b}^N$ is defined by

(V.6) $\forall\, \varphi, \psi \in X^N\ ,\ \hat{b}^N(\varphi,\psi) = \sum_{n=0}^N A_{nn} \varphi(\xi_n^N) \psi(\xi_n^N)\, \rho_n^N$ ,

where $A_{nn}$ corresponds to the diagonal entry of the stiffness matrix A, equal to

$$A_{nn} = a^N(h_n^N,\, h_n^N)\ ,$$

and we recall that $h_n^N$ is the Lagrangian interpolant at the point $\xi_n^N$ .From hypothesis (V.2) and the exactness of the Gauss-Lobatto quadrature formula, it is simple to derive that

(V.7) $\forall\ \varphi \in X^N\ ,\ a^N(\varphi,\varphi) \leqslant \alpha^+\, a(\varphi,\varphi) \leqslant \alpha^+\, N(N-1)\, \tilde{b}(\varphi,\varphi)$

$$\leqslant (3\alpha^+/\alpha^-)(N-1)/(N+1)\ b^N(\varphi,\varphi),$$

where we recall that $\tilde{b}(\varphi,\psi) = \int_\Lambda \varphi(x)\ \psi(x)/(1-x^2)\ dx$. As a result, we derive that the normalization of $b^N$ satifies

(V.8) $b^N = (3\alpha^+/\alpha^-)(N-1)/(N+1)\ \tilde{b}^N$ .

From the eigenvalue problem : Find $\Psi$ in $X^N$ and $\lambda$ in $\mathbb{R}^+$ such that

(V.9) $\forall\ v \in X^N\ ,\ a^N(\Psi,v) = \lambda\, b^N(\Psi,v)$

that possess $N-1$ eigenvalues $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_{N-1} \leqslant 1$, the eigenvectors of which are chosen normalized with respect to the norm derived from $b^N$ we define, as in (II.10) the $|||\ .\ |||_{\theta,N}$ norms for any $\theta \in \mathbb{R}$ by

(V.10) $\forall\ v \in X^N\ ,\ |||\ v\ |||^2_{\theta,N} = \sum_{n=1}^{N-1} \lambda_n^\theta\, b^N(v,\Psi_n)^2$.

Here the bilinear forms associated to the definition of the system and to the smoothing depend on N. The definitions of the smoothing operator $\mathcal{S}^N$ and the correction operator $\mathcal{C}^N$ have to be precised. Let us do this in the simple case where only two grids are used. Instead of (II.3), the smoothing procedure consists in : Find $\mathcal{S}^N\varphi$ in $X^N$ such that

(V.11) $\forall\ v \in X^N\ ,\ b^N\,(\mathcal{S}^N\varphi-\varphi,v) = g^N(v)-a^N(\varphi,v)$ ,

while the correction procedure consists in : Find $\bar{\varphi}$ in $X^{N/2}$ such that

(V.12) $\forall\ v \in X^{N/2}\ ,\ a^{N/2}(\bar{\varphi},v) = g^N(v)-a^N(\varphi,v)$ ,

and define $\mathcal{C}^N\varphi = \varphi + \bar{\varphi}$. Then as explained in the general case, during a V-cycle with $m/2$ smoothings down and up the error $e^0$ is changed in $e^1$ as follows

(V.13) $e^1 = (\mathcal{S}^N)^{m/2}\, \mathcal{C}^N\, (\mathcal{S}^N)^{m/2}\, e^0$ .

Before entering in the details of the analysis of the decay rate of the error, let us state some results of general interest. First of, let $\chi$ be in $\mathbb{P}_n^0(\Lambda)$ be given, $r(\chi)$ and $r_N(\chi)$ in $\mathbb{P}_N^0(\Lambda)$ be such that

(V.14) $\forall\ \psi \in \mathbb{P}_N^0(\Lambda)\ ,\ b(r(\chi),\psi) = b^N(r_N(\chi),\psi) = a^N(\chi,\psi)$ ,

where b is defined in (III.13). It is a simple consequence of (V.9) to derive that

$$|||\ r_N(\chi)\ |||^2_{0,N} = b^N(r_N(\chi),r_N(\chi)) = a^N(\chi,r_N(\chi)) \leqslant |||\ \chi\ |||_{2,N}\ |||\ r_N(\chi)\ |||_{0,N}\ ,$$

whence

$$|||\ r_N(\chi)\ |||_{0,N} \leqslant |||\ \chi\ |||_{2,N}\ .$$

Besides, from (V.7) and (V.14), we have

$$b(r(\chi),r(\chi)) = b^N(r_N(\chi),r(\chi)) \leq [b^N(r_N(\chi),r_N(\chi))\, b^N(r(\chi),r(\chi))]^{1/2}$$
$$\leq (\alpha^{+2}/\alpha^{-})^{1/2}\, [b^N(r_N(\chi),r_N(\chi))\, b(r(\chi),r(\chi))]^{1/2} \ ;$$

finally, we derive that the solution $r(\chi)$ of (V.14) has the following stability property

(V.15) $\quad b(r(\chi),r(\chi)) \leq c\, |||\, \chi\, |||_{2,N}$ .

Then let us state some approximation result the proof of which will be presented in the appendix.

Lemma V.1 : *Let* $\varrho$ *defined over* $\Lambda$ *and such that*

(V.16) $\quad \int_\Lambda \varrho^2(x)/(1-x^2)\, dx < \infty$ ,

*and* $n$ *be a positive integer. The solutions* $\varphi(\varrho) \in H_0^1(\Lambda)$, $\varphi_n(\varrho) \in \mathbb{P}_n^0(\Lambda)$ *defined by*

(V.17) $\quad \forall\, v \in H_0^1(\Lambda)\ ,\ a(\varphi(\varrho),v) = \tilde{b}(\varrho,v)$ ,

(V.18) $\quad \forall\, v \in \mathbb{P}_n^0(\Lambda)\ ,\ a^n(\varphi_n(\varrho),v) = \tilde{b}(\varrho,v)$ ;

*then the following approximation results holds*

(V.19) $\quad |\, \varphi(\varrho) - \varphi_n(\varrho)\, |_1^2 \leq c\, n^{-2}\, \tilde{b}(\varrho,\varrho)$ ,

(V.20) $\quad \tilde{b}(\varphi(\varrho) - \varphi_n(\varrho), \varphi(\varrho) - \varphi_n(\varrho)) \leq c\, n^{-4}\, \tilde{b}(\varrho,\varrho)$ .

Let us denote by $\varepsilon$ the term $(\mathcal{S}^N)^{m/2} e^0$ ; derive now as in (II.12) that,

(V.21) $\quad |||\, e^1\, |||_{1,N} \leq f(m/2,1/2)\, |||\, \mathcal{C}^N \varepsilon\, |||_{0,N}$ ,

(V.22) $\quad |||\, \varepsilon\, |||_{2,N} \leq f(m/2,1/2)\, |||\, e^0\, |||_{1,N}$ .

From the definition of $\mathcal{C}^N$, we derive that, for any $\chi$ in $\mathbb{P}_N^0(\Lambda)$

$$|||\, \mathcal{C}^N \chi\, |||_{0,N} = |||\, \chi - \bar{\chi}\, |||_{0,N} \leq |||\, \chi - N(N-1)\varphi(r(\chi))\, |||_{0,N} + |||\, \bar{\chi} - N(N-1)\varphi(r(\chi))\, |||_{0,N} \ ;$$

hence

$$|||\, \mathcal{C}^N \chi\, |||_{0,N} \leq (\alpha^{+2}/\alpha^{-})[\, b(\chi - N(N-1)\varphi(r(\chi)),\, \chi - N(N-1)\varphi(r(\chi))^{1/2}$$
$$+ b(\bar{\chi} - N(N-1)\varphi(r(\chi)),\, \bar{\chi} - N(N-1)\varphi(r(\chi))^{1/2}\, ] \ .$$

With the previous notations (see (V.14) and (V.18)) and recalling that the normalization factor between the forms $b$ and $\tilde{b}$ is $N(N-1)$ (see (III.14)), it is simple to derive that $\chi = N(N-1)\varphi_N(r(\chi))$, while $\bar{\chi}$ (defined in (V.12)) satisfies $\bar{\chi} = N(N-1)\varphi_{N/2}(r(\chi))$; we derive

$$|||\, \mathcal{C}^N \chi\, |||_{0,N} \leq (\alpha^{+2}/\alpha^{-})(N(N-1))[\, b(\varphi_N(r(\chi)) - \varphi(r(\chi)),\, \varphi_N(r(\chi)) - \varphi(r(\chi))^{1/2}$$
$$+ b(\, \varphi_{N/2}(r(\chi)) - \varphi(r(\chi)),\, \varphi_{N/2}(r(\chi)) - \varphi(r(\chi))^{1/2}\, ] \ ,$$

applying next (V.20) for $n = N$ and $n = N/2$, then (V.15), we obtain

Lemma V.2 : *There exists a constant* c *independent of both* N *and* K *such that*

(V.23) $\forall\ \chi \in \mathbb{P}_N^0(\Lambda)\ ,\ |||\ \mathcal{C}^N \chi\ |||_{0,N} \leq c\ |||\ \chi\ |||_{2,N}$ .

Plugging (V.23) in (V.21) and using (V.22) give now the following

Theorem V.1 : *The multigrid algorithm based on the Jacobi preconditioner converges, and at any* V-cycle *with* m *smoothings as detailed in section* II , *the rate of convergence is given by*

(V.24) $\int_\Lambda e_1'^2(x)\,dx \leq c\ f(m,1) \int_\Lambda e_0'^2(x)\,dx$ .

Remark V.1 : The decay rate of multigrid algorithm is independent of N, and it is interesting to note that its asymptotic behavior is $\mathcal{O}(1/m)$.

Remark V.2 : The analysis of the case of non constant coefficients requires some regularity (i.e., $W^{2,\infty}$ ) of the coefficient ( this is required for lemma V.1). We do not know to what extend the lack of regularity of $\alpha$ deteriorates the actual convergence. Note however that the local regularity is just required, i.e., $\alpha$ can be non smooth through the interfaces. The multigrid procedure proposed in [ZWH1.2], and that is based on another approach of smoothing, is numerically proved to be very robust with respect to the irregularity of $\alpha$ [*].

## A. APPENDIX

The main purpose of this appendix is to provide the proof of lemma V.1. Let us first recall the following result of the approximation theory (see [D])

Lemma A.1 : *Let* $\pi_M$ *denote the orthogonal projection operator from* $L^2(\Lambda)$ *onto* $\mathbb{P}_M(\Lambda)$. *The following approximation results hold*

(A.1) $\forall\ u \in H^m(\Lambda)\ ,\ \|\ u - \pi_M u\ \|_0 \leq c(m)\ M^{-m}\ [\ \int_\Lambda u^{(m)2}(x)\ (1-x^2)^m\,dx]^{1/2}$ .

**Proof** : Let us recall that the Legendre polynomials constitute a total system of orthogonal functions of $L^2(\Lambda)$ that verifies

(A.2) $\int_\Lambda L_n(x)\ L_m(x)\,dx = \delta_{nm}\ 2/(2n+1)$ .

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Let us use this basis to span the function u; we arrive at

$$u = \sum_{n=0}^{\infty} a_n L_n \ ,$$

so that the $L^2(\Lambda)$-projection of u onto $\mathbb{P}_M(\Lambda)$ is equal to

$$\pi_M u = \sum_{n=0}^{M} a_n L_n \ ,$$

and the error

$$u - \pi_M u = \sum_{n=M+1}^{\infty} a_n L_n \ . \tag{A.3}$$

Let us recall now that the Legendre polynomials satisfy the following relation

$$((1-x^2) L_n')' = - n(n+1) L_n \ .$$

From (A.2) we conclude that

$$\int_\Lambda L_n'(x) L_m'(x) (1-x^2) dx = \delta_{nm} 2n(n+1)/(2n+1) \ . \tag{A.4}$$

From (A.2) and (A.3) we get

$$\| u - \pi_M u \|_0 = \sum_{n=M+1}^{\infty} 2 a_n^2 /(2n+1) \ ,$$

while, from (A.4), we derive that

$$\int_\Lambda u'^2(x) (1-x^2) dx = \sum_{n=0}^{\infty} 2n(n+1) a_n^2 /(2n+1) \ ,$$

and (A.1) is then just a simple consequence of these two equalities in the case m = 1. The general case is handled by recursion.

As a consequence, we derive that

Corollary A.1 : *Let* $\pi_M^1$ *denote the orthogonal projection operator from* $H_0^1(\Lambda)$ *onto* $\mathbb{P}_M^0(\Lambda)$. *The following approximation results hold*

$$\forall\, u \in H^m(\Lambda) \cap H_0^1(\Lambda) \ , \ \| u - \pi_M^1 u \|_1 \leq c(m) M^{-m} \left[ \int_\Lambda u^{(m+1)2}(x) (1-x^2)^m dx \right]^{1/2} \ . \tag{A.5}$$

Proof : It has already been noted that for any u in $H_0^1(\Lambda)$

$$[\pi_M^1 u](x) = \int_{-1}^{x} \pi_{M-1}(u')(t) dt \ ,$$

so that (A.5) is an easy consequence of lemma A.1.

**Proof of lemma V.1** : It is an easy matter to note that the solution $\varphi(\varrho)$ of problem (V.17) satisfies

$$\forall\, x \in \Lambda \ , \ - [(\alpha[\varphi(\varrho)]_x)_x](x) = \varrho(x)/(1-x^2) \ .$$

From (V.2) and (V.16), we derive

$$\int_\Lambda [\varphi(\varrho)]_x^2(x) dx + \int_\Lambda [\varphi(\varrho)]_{xx}^2(x) (1-x^2) dx \leq c \int_\Lambda \varrho^2(x)/(1-x^2) dx \leq c \tilde{b}(\varrho,\varrho) \ . \tag{A.6}$$

From corollary A.1, we know that there exists an element $\psi_n$ of $\mathbb{P}_{n/2}^0(\Lambda)$, such that

$$| \varphi(\varrho) - \psi_n |_1 \leq c\, n^{-1} \int_\Lambda [\varphi(\varrho)]''_{xx}(x)\,(1-x^2)\,dx \ ,$$

and

$$| \psi_n |_1 \leq | \varphi(\varrho) |_1 \ ,$$

so that

(A.7) $$| \varphi(\varrho) - \psi_n |_1 \leq c\, n^{-1}\, \tilde{b}(\varrho,\varrho) \ .$$

From the ellipticity of the form $a^n$, we derive that there exists a constant $\beta$, such that

$$\beta\, | \varphi_n(\varrho) - \psi_n |_1^2 \leq a^n(\varphi_n(\varrho) - \psi_n \ , \varphi_n(\varrho) - \psi_n) \ ;$$

using (V.17) and (V.18) we derive that

$$\beta\, | \varphi_n(\varrho) - \psi_n |_1^2 \leq a(\varphi(\varrho), \varphi_n(\varrho) - \psi_n) - a^n(\psi_n \ , \varphi_n(\varrho) - \psi_n) \ ,$$

or again

(A.8) $$\beta\, | \varphi_n(\varrho) - \psi_n |_1^2 \leq a(\varphi(\varrho) - \psi_n \ , \varphi_n(\varrho) - \psi_n) + a(\psi_n \ , \varphi_n(\varrho) - \psi_n) - a^n(\psi_n \ , \varphi_n(\varrho) - \psi_n) \ .$$

Let us note now that from the exactness of the Gauss Lobatto quadrature formula we get that

(A.9) $$\forall\, \alpha_n \in \mathbb{P}_{n/2}(\Lambda),\ (\alpha_n \psi_n' \ , (\varphi_n(\varrho) - \psi_n)') = (\alpha_n \psi_n' \ , (\varphi_n(\varrho) - \psi_n)')_{n,GL} \ ,$$

so that

$$a(\psi_n \ , \varphi_n(\varrho) - \psi_n) - a^n(\psi_n \ , \varphi_n(\varrho) - \psi_n) = ((\alpha - \alpha_n)\psi_n' \ , (\varphi_n(\varrho) - \psi_n)) - ((\alpha - \alpha_n)\psi_n' \ , (\varphi_n(\varrho) - \psi_n))_n \ .$$

and using now (A.8) yields

(A.10) $$| \varphi_n(\varrho) - \psi_n |_1 \leq c\, (\, | \varphi(\varrho) - \psi_n |_1 \ + | \alpha - \alpha_n |_{L^\infty(\Lambda)}\, | \psi_n |_1 \,) \ .$$

It is standard to note that there exists an element $\alpha_n$ such that

$$| \alpha - \alpha_n |_{L^\infty(\Lambda)} \leq c\, n^{-k} \sum_{j=0}^{k} | \alpha^{(j)} |_{L^\infty(\Lambda)} \ ;$$

hence, from (A.10) and (A.7), we derive that

$$| \varphi(\varrho) - \varphi_n(\varrho) |_1 \leq c\, n^{-1}\, (b(\varrho,\varrho)^{1/2} + (\textstyle\sum_{j=0}^{2} | \alpha^{(j)} |_{L^\infty(\Lambda)})\ | \varphi(\varrho) |_1 \,) \ ,$$

and (V.19) is an easy consequence of (A.6).

Let us turn now to the proof of (V.20). We shall use a standard duality technique and first define an element $\chi$ in $H_0^1(\Lambda)$ as follows

(A.11) $$\forall\, v \in H_0^1(\Lambda) \ , \ a(\chi, v) = b(\varphi(\varrho) - \varphi_n(\varrho), v) \ .$$

It is an easy matter to derive

$$\begin{aligned} b(\varphi(\varrho) - \varphi_n(\varrho), \varphi(\varrho) - \varphi_n(\varrho)) &= a(\chi, \varphi(\varrho) - \varphi_n(\varrho)) \\ &= a(\chi - \pi^1_{n/2}\chi \ , \varphi(\varrho) - \varphi_n(\varrho)) + a(\pi^1_{n/2}\chi \ , \varphi(\varrho) - \varphi_n(\varrho)) \\ &= a(\chi - \pi^1_{n/2}\chi \ , \varphi(\varrho) - \varphi_n(\varrho)) + (a^n - a)(\pi^1_{n/2}\chi \ , \varphi_n(\varrho)) \ . \end{aligned}$$

Let us examine the last term on the right-hand side of this last equality; using (A.9) one more time

we write

$$(a^n - a)(\pi^1_{n/2}\,\chi\,,\,\varphi_n(\varrho)) = ((\alpha - \alpha_n)(\,\pi^1_{n/2}\,\chi)',\varphi_n(\varrho)')_n + ((\alpha - \alpha_n)(\,\pi^1_{n/2}\,\chi)',\varphi_n(\varrho)')\quad,$$

so that

$$b(\varphi(\varrho) - \varphi_n(\varrho),\varphi(\varrho) - \varphi_n(\varrho)) = a(\chi - \pi^1_{n/2}\,\chi,\varphi(\varrho) - \varphi_n(\varrho)) + ((\alpha - \alpha_n)(\pi^1_{n/2}\,\chi)',\varphi_n(\varrho)')_n + ((\alpha - \alpha_n)(\,\pi^1_{n/2}\,\chi)',\varphi_n(\varrho)')\quad;$$

hence

$$\begin{aligned} b(\varphi(\varrho) - \varphi_n(\varrho),\varphi(\varrho) - \varphi_n(\varrho)) &\leqslant c\,|\,\chi - \pi^1_{n/2}\,\chi\,|_1\,|\,\varphi(\varrho) - \varphi_n(\varrho)\,|_1 \\ &\quad + c\,n^{-2}\sum_{j=0}^{2}|\,\alpha^{(j)}\,|_{L^\infty(\Lambda)}\,|\,\pi^1_{n/2}\,\chi\,|_1\,|\,\varphi_n(\varrho)\,|_1\ , \\ &\leqslant c\,|\,\chi - \pi^1_{n/2}\,\chi\,|_1\,|\,\varphi(\varrho) - \varphi_n(\varrho)\,|_1 \\ &\quad + c\,n^{-2}\sum_{j=0}^{2}|\,\alpha^{(j)}\,|_{L^\infty(\Lambda)}\,|\,\pi^1_{n/2}\,\chi\,|_1\,|\,\varphi_n(\varrho)\,|_1\ . \end{aligned}$$

Using now corollary A.1 and (A.11), we derive (after bounding $\sum_{j=0}^{2}|\,\alpha^{(j)}\,|_{L^\infty(\Lambda)}$ by a constant)

$$[b(\varphi(\varrho) - \varphi_n(\varrho),\varphi(\varrho) - \varphi_n(\varrho))]^{1/2} \leqslant c\,(n^{-1}\,|\varphi(\varrho) - \varphi_n(\varrho)|_1 + n^{-2}\,|\,\varphi_n(\varrho)\,|_1)\ ;$$

thanks to the stability of $\varphi_n(\varrho)$ with respect to $\varrho$, we derive (V.20) from (V.19).

**Acknowledgements**: The authors would like to thank Prof. A. T. Patera for interesting discussions along the elaboration of this paper and Prof. R. Verfurth for worthy ideas concerning the proof of section V.

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# Report Documentation Page

| 1. Report No. | 2. Government Accession No. | 3. Recipient's Catalog No. |
|---|---|---|
| NASA CR-181761<br>ICASE Report No. 88-73 | | |

| 4. Title and Subtitle | 5. Report Date |
|---|---|
| SPECTRAL ELEMENT MULTIGRID<br>PART 2: THEORETICAL JUSTIFICATION | December 1988 |
| | 6. Performing Organization Code |

| 7. Author(s) | 8. Performing Organization Report No. |
|---|---|
| Yvon Maday and Rafael Munoz | 88-73 |
| | 10. Work Unit No. 505-90-21-01 |

| 9. Performing Organization Name and Address | 11. Contract or Grant No. |
|---|---|
| Institute for Computer Applications in Science and Engineering<br>Mail Stop 132C, NASA Langley Research Center<br>Hampton, VA 23665-5225 | NAS1-18107, NAS1-18605 |
| | 13. Type of Report and Period Covered: Contractor Report |
| 12. Sponsoring Agency Name and Address<br>National Aeronautics and Space Administration<br>Langley Research Center<br>Hampton, VA 23665-5225 | 14. Sponsoring Agency Code |

15. Supplementary Notes

Langley Technical Monitor: Richard W. Barnwell

Final Report

Submitted to J. Sci. Comput.

16. Abstract

We analyze here a multigrid algorithm used for solving iteratively the algebraic system resulting from the approximation of a second order problem by spectral or spectral element methods. The analysis, performed here in the one dimensional case, justifies the good smoothing properties of the Jacobi preconditioner, that has been presented in part 1 of this paper.

Keywords:

| 17. Key Words (Suggested by Author(s)) | 18. Distribution Statement |
|---|---|
| spectral, spectral element, multigrid; | 64 Numerical Analysis, (j 10)<br>Unclassified - unlimited |

| 19. Security Classif. (of this report) | 20. Security Classif. (of this page) | 21. No. of pages | 22. Price |
|---|---|---|---|
| Unclassified | Unclassified | 30 | A03 |

NASA FORM 1626 OCT 86

NASA-Langley, 1988